Input-to-state stability and averaging of linear fast switching systems

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Abstract

We consider the averaging method for stability of rapidly switching linear systems with disturbances. We show that the notions of strong and weak averages proposed in [1], with partial strong average defined in this paper, play an important role in the context of switched systems. Using these notions of average, we show that exponential ISS of the strong and the partial strong average system with linear gain imply exponential ISS with linear gain of the actual system. Similarly, exponential ISS of the weak average guarantees an appropriate exponential derivative ISS (DISS) property for the actual system. Moreover, using the Lyapunov method, we show that linear ISS gains of the actual system and its average converge to each other as the switching rate is increased.

1 Introduction

Switched systems have been used extensively in various areas of control engineering, such as mechanical systems, automotive industry, aircraft control and power electronics [2, 3]. Switched systems are dynamical systems governed by differential equations whose right hand side is selected from a given family of functions based on some switching rule. The stability properties of the systems with fast switching behavior have been considered recently, where averaging plays an important role. These averaging results were used in applications of such as network stability analysis, synchronization of chaotic oscillators and control of multiple autonomous agents [4–6]. In particular, [5, 6] investigate the exponential stability of fast switching linear switched systems via averaging, and the same authors analyze the finite $\mathcal{L}_2$ gain for systems with inputs [7].

Our results are closely related to the recent averaging techniques for continuous-time systems with disturbances in [1] where ISS is investigated. Although results in [1] are not written to deal exclusively with switched systems, they are general enough to include as a special case nonlinear switched systems. However, their conclusions are too weak whenever the ISS disturbance gain is linear and the decay of transients is exponential. Such situation often arises in linear switched systems and, hence, there is a strong motivation for sharpening the results in [1] to this important situation.

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With the notions of strong and weak averages pioneered in [1] and partial strong average that we propose here, we derive main results when the average system is ISS with an exponential $\mathcal{KL}$ estimate and a linear gain. We show that exponential ISS of the strong average implies exponential ISS with linear gain for the actual linear switched system for sufficiently high switching rates. We also show that if the weak average is exponentially ISS with linear gain, then the actual linear switched system satisfies an exponential DISS property. In addition, with partial averaging that has been used to study the stability properties for continuous time varying nonlinear systems [8], we present stronger conclusions when there does not exist a strong average. The partial strong average is defined in the present paper to show that its exponential ISS implies exponential ISS properties for the actual system when switching is fast enough. Moreover, based on the Lyapunov method, we show that the linear ISS gain of the actual system converges to the ISS gain of its average as the switching rate is increased. These new results provide novel insights on robustness in the context of linear switched systems, and we believe that these average notions will play an important role in future developments of averaging methodology for switched systems with disturbances.

The paper is organized as follows. Definitions and preliminary results are presented in Section 2. Sections 3 and 4 contain main results and their application to the cascaded systems. A summary is given in the last section and the proofs are presented in the Appendix.

2 Preliminaries

A function $\tilde{\gamma} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is zero at zero, continuous and strictly increasing. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{L}$ if it is converging to zero as its argument grows unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if it is of class-$\mathcal{K}$ in its first argument, and class-$\mathcal{L}$ in its second argument, and a class-$\mathcal{KL}$ function $\beta(\cdot, s)$ is called exponential if $\beta(r, s) = Kr\exp(-\lambda s)$ for some $K > 0$, $\lambda > 0$. $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ are the minimum and the maximum eigenvalue of a matrix, respectively. Given a measurable function w, we define its infinity norm as $||w||_\infty := \text{ess } \sup_{t\geq 0} |w(t)|$. If we have $||w||_\infty < \infty$, then we write $w \in \mathcal{L}_\infty$. $|\cdot|$ denotes the vector norm.

We consider linear fast switching systems that depend on a small parameter $\varepsilon > 0$,

$$\dot{x} = A_{\rho(\frac{t}{\varepsilon})}x + B_{\rho(\frac{t}{\varepsilon})}w \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^m$ is the input, (A_i, B_i) is a family of constant matrices that is parameterized by some index $i \in S \triangleq \{1, 2, \cdots, N\}$, $\rho : \mathbb{R}_{\geq 0} \to S$ is a piecewise constant function of time, called a switching signal. Given $\rho\left(\frac{t}{\varepsilon}\right)$, suppose that for every ε there exists $\nu := \nu(\varepsilon) > 0$ such that the interval between consecutive switching times is not smaller than ν. Note that the switching rate increases as the parameter ε decreases, but for any fixed and arbitrarily small ε, the above assumption guarantees that we do not have Zeno solutions.

We also consider a more general class of linear fast switching systems

$$\dot{x} = A_{\rho_1(\frac{t}{\varepsilon})}x + B_{\rho_2(t,\varepsilon)}w, \tag{2}$$

where ρ_2 is a function of (t, ε) with the form $\frac{t}{\varepsilon}$ being a special case. We first recall the definition of weak and strong averages for nonlinear systems $\dot{x} = f(t, x, w)$ (introduced in [1]) and define the partial strong average for the system $\dot{x} = f_p(\frac{t}{\varepsilon}, t, x, w, \varepsilon)$. Then, in Definition 4, we will introduce the notion of average for switched matrices and use it to generate various type of averages in Definitions 1-3 for linear switched systems (1).

Definition 1 *(Weak Average) A locally Lipschitz function $f_{wa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be a weak average of f if there exists $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ such that $\forall\ t \geq 0$, $\forall\ T \geq T^*$, $\forall\ w \in \mathbb{R}^m$, and $\forall\ x \in \mathbb{R}^n$, the following holds:*

$$\left| f_{wa}(x, w) - \frac{1}{T}\int_t^{t+T} f(s, x, w)ds \right| \leq \beta_{av}(\max\{|x|, |w|, 1\}, T). \tag{3}$$

Definition 2 *(Strong Average) A locally Lipschitz function $f_{sa} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be a strong average of f if there exists $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ such that $\forall\ t \geq 0$, $\forall\ T \geq T^*$, $\forall\ w \in \mathcal{L}_\infty$, and $\forall\ x \in \mathbb{R}^n$, the following holds:*

$$\left| \frac{1}{T}\int_t^{t+T} [f_{sa}(x, w(s)) - f(s, x, w(s))]ds \right| \leq \beta_{av}(\max\{|x|, ||w||_\infty, 1\}, T). \tag{4}$$

Definition 3 *(Partial Strong Average) A locally Lipschitz function $f_{psa} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be a partial strong average of f_p if there exists $\beta_{av} \in \mathcal{KL}$ and $T^* > 0$ such that $\forall\ t \geq 0$, $\forall\ T \geq T^*$, $\forall\ w \in \mathcal{L}_\infty$, and $\forall\ x \in \mathbb{R}^n$, the following holds:*

$$\left| \frac{1}{T}\int_t^{t+T} \left[f_{psa}(t, x, w(s)) - f_p\left(\frac{s}{\varepsilon}, t, x, w(s), \varepsilon\right)\right] ds \right| \leq \beta_{av}(\max\{|x|, ||w||_\infty, 1\}, T). \tag{5}$$

Remark 1 *Note that the main difference between the weak and the strong average is that in the definition of weak average the disturbance is kept constant in (3) whereas for strong average the inequality (4) needs to hold for all disturbances $w \in \mathcal{L}_\infty$. A complete characterization of strong averages for continuous-time systems was given in [1], where it is shown that any $f(t, x, w)$ that is periodic in t has a strong average if and only if the function f has the structure as $f(t, x, w) = f_1(t, x) + f_2(x, w)$ and there exists the average $f_{av}(x)$ for $f_1(t, x)$ according to either strong or weak average definition (they coincide as f_1 does not depend on disturbances). Then, $f_{sa}(x, w) := f_{av}(x) + f_2(x, w)$ satisfies* *the strong average definition.*

We next present, the definition of average for switched matrices that is used to define strong, partial strong and weak average systems for the linear switched system (1).

Definition 4 *(Average for switched matrices) A constant matrix A_{av} is said to be an average of $A_{\rho(t)}$ if for the switching function ρ, there exist a class-$\mathcal{L}$ function σ and positive real numbers k and T^*, such that $\forall t \geq 0$, $\forall T \geq T^*$, the following holds:*

$$\left| A_{av} - \frac{1}{T}\int_t^{t+T} A_{\rho(s)} ds \right| \leq k\sigma(T). \tag{6}$$

Remark 2 *Note that the average of switched matrices does not necessarily imply that $\rho(\cdot) : \mathbb{R}_{\geq 0} \to S$ is periodic. On the other hand, suppose that $\rho(\cdot)$ is periodic of period T. It can be shown that the average matrix defined as $A_{av} = \frac{1}{T}\int_0^T A_{\rho(t)} dt$ satisfies Definition 4 for some k and σ. Let T_i be the length of time during one period for which $\rho(t) = i$. Then, it is not hard to see that $A_{av} = \frac{1}{T}\sum_i A_i T_i = \sum_i \lambda_i A_i$, where by definition $\lambda_i = T_i/T$ and $\sum_i \lambda_i = 1$.*

Using the definition of averages for switched matrices, we will concentrate on the strong and the weak averages for system (1) and the partial strong average for system (2). In particular, it can be shown that if A_{av} and B_{av} are respectively averages of $A_{\rho(t)}$ and $B_{\rho(t)}$ in (1) under Definition 4, the system $\dot{x} = A_{av}x + B_{av}w$ satisfies the weak average definition. On the other hand, if we have that $B_i = B$ for all $i \in \{1, 2, \cdots, N\}$, then the system $\dot{x} = A_{av}x + Bw$ satisfies the definition of strong average in Definition 2. Finally, we will also consider the system $\dot{x} = A_{av}x + B_{\rho_2(t,\varepsilon)}w$. It is not hard to show that this system is a partial strong average of system (2) under Definition 3.

Remark 3 *All the notions of strong, partial strong and weak averages are useful in different situations and we investigate all of them. It was shown in [1] for continuous-time systems that strong averages exist for a smaller class of systems but using them we can state stronger stability results (see Theorem 2), while weak averages exist for a larger class of systems but using them we can state weaker stability results (see Theorem 1). In particular, weak averages are found useful when one deals with ISS of cascaded systems (see Section 4). The definition of partial strong average is not considered in [1], and it appears to be novel in the context of switched systems. However, it can apply to more general systems (2) when there does not exist an average for $B_{\rho_2(t,\varepsilon)}$. Moreover, noting that system (1) is a special case of system (2), we show in Theorem 3 that the notion of partial strong average is useful to conclude ISS for system (1) in cases when a strong average does not exist and weak average would give too weak conclusions (i.e. DISS).*

From Remark 1, we know that existence of the strong average always implies existence of weak average. The opposite does not hold as the following example illustrates.

Example 1 *Consider the linear switched system (1). The switching law $\rho(\frac{t}{\varepsilon})$ selects elements of the set $S = \{1, 2\}$ according to the policy*

$$\rho\left(\frac{t}{\varepsilon}\right) = \begin{cases} 1 & \frac{t}{\varepsilon} \in \left[n\pi, \frac{(2n+1)\pi}{2}\right) \\ 2 & \frac{t}{\varepsilon} \in \left[\frac{(2n+1)\pi}{2}, (n+1)\pi\right), \end{cases}$$

where $n \in \mathbb{N}_{\geq 0}$. Let $A_1 = A_2 := -0.5$, $B_1 = 1$, and $B_2 = -1$. From Definition 1 and Remark 2, we know that the weak average of this system is $\dot{y} = -0.5y$.

Now, we will show that there does not exist a strong average for system (1). Pick an arbitrary $\bar{x} \neq 0$ and note that, for any given function $f_{sa}(x, w)$, we have two possibilities:
***a.** either $f_{sa}(\bar{x}, w) + 0.5\bar{x} = 0$, $\forall w$, or **b.** $\exists \bar{w}$, such that $f_{sa}(\bar{x}, \bar{w}) + 0.5\bar{x} \neq 0$.*
*Suppose f_{sa} is the strong average for system (1) and case **a** holds. Let $w(t) = B_{\rho(t)}$, we have*

$$\left|\frac{1}{T}\int_t^{t+T} \left(B_{\rho(s)}\right)^2 ds\right| = \left|\frac{1}{T}\int_t^{t+T} 1 \; ds\right| = 1 \quad \forall T > 0.$$

*Suppose that f_{sa} is the strong average for system (1) and case **b** holds. Pick $w(t) = \bar{w}$ and set $T = C\pi$ for $C \in \mathbb{N}$, through simple calculation one gets*

$$\left|\frac{1}{T}\int_t^{t+T} \{f_{sa}(\bar{x}, \bar{w}) + 0.5\bar{x} - B_{\rho(s)}\bar{w}\}ds\right| = |f_{sa}(\bar{x}, \bar{w}) + 0.5\bar{x}| > 0 \quad \forall C > 0,$$

which does not converge to zero as C approaches infinity, in other words as $T \to \infty$. Thus, there does not exist a strong average for system (1).

In order to state our main results, we need the definition of exponential-ISS with linear disturbance gain (see Section 4.9 [9]).

Definition 5 *The system $\dot{x} = f_a(t, x, w)$ is said to be exponentially ISS with linear gain γ if there exist positive constants K, λ such that, for all $t_0 \in \mathbb{R}_{\geq 0}$, $w \in \mathcal{L}_\infty$ and each $x_0 := x(t_0) \in \mathbb{R}^n$, the solution of the system starting at (x_0, t_0) exists for all $t \geq t_0$ and satisfies*

$$|x(t)| \leq K \exp(-\lambda(t - t_0))|x_0| + \gamma||w||_\infty, \forall t \geq t_0 \geq 0. \tag{7}$$

3 Main Results

We next present the main results that can be used to conclude exponential-DISS and exponential-ISS of system (1) via its weak, strong and partial strong averages. The proofs of Theorem 2 and 3 are omitted as they are nearly identical to the proof of Theorem 1. The proof of Theorem 1 and the technical lemma are given in the appendix.

Assumption 1 *Consider the weak (strong) average* $\dot{y} = A_{av}y + B_{av}w$ *to system (1), suppose there exist* $\gamma_a > 0$ *and a symmetric positive definite constant matrix* P *such that there exist positive real numbers* c_1*,* c_2*, and the quadratic Lyapunov function* $V = y^T P y$ *satisfying*

$$c_1|y|^2 \leq V(y) \leq c_2|y|^2, \qquad \frac{dV}{dy}(y)\,(Ay + Bw) \leq -|y|^2 + \gamma_a|w|^2 \quad \forall y, w. \tag{8}$$

Remark 4 *If Assumptions 1 holds, then the system* $\dot{y} = A_{av}y + B_{av}w$ *is exponentially ISS. From [10], for a given quadratic Lyapunov function* V *there exist positive real numbers* K, λ *and* γ *satisfying* $K = \sqrt{\frac{c_2}{c_1}}$*,* $\lambda = \frac{1}{2c_2}$ *and* $\gamma = \sqrt{\frac{c_2\gamma_a}{c_1}}$ *such that (7) holds. Note in Assumption 1, the constant matrix* P *can be calculated through the Lyapunov matrix equation* $A_{av}^T P + PA_{av} = -I$*, and then* $c_1 = \lambda_{min}(P)$ *and* $c_2 = \lambda_{max}(P)$*.*

Theorem 1 *Suppose that the weak average of system (1) exists and satisfies Assumption 1. Then, for any* $\delta > 0$ *there exists* $\varepsilon^* > 0$ *and for all* $\varepsilon \in (0, \varepsilon^*)$*,* $w, \dot{w} \in \mathcal{L}_\infty$ *and* $x_0 := x(t_0) \in \mathbb{R}^n$*, the solution of system (1) satisfies:*

$$|x(t)| \leq (K + \delta)\exp(-(\lambda - \delta)(t - t_0))|x_0| + (\gamma + \delta)||w||_\infty + \delta||\dot{w}||_\infty \qquad \forall t \geq t_0 \geq 0. \tag{9}$$

where positive constants K, λ, γ *come from Remark 4. Thus, the system (1) is exponentially derivative input-to-state stable (DISS) uniformly in small* ε*.*

Remark 5 *The results in Theorem 1 can be defined as a stronger exponential version of the definition of DISS in [11]. The system* $\dot{x} = f_a(t, x, w)$ *is said to be DISS if there exists* $\beta \in \mathcal{KL}$*, and some class-*$\mathcal{K}$ *functions* $\tilde{\gamma}_0$ *and* $\tilde{\gamma}_1$ *such that, for each* $w, \dot{w} \in \mathcal{L}_\infty$ *and each* $x_0 := x(t_0) \in \mathbb{R}^n$*, solutions of the system starting at* (x_0, t_0) *exists for all* $t \geq t_0$ *and satisfies*

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \tilde{\gamma}_0(||w||) + \tilde{\gamma}_1(||\dot{w}||).$$

Note that our conclusion in Theorem 1 yields a stronger condition, where $\tilde{\gamma}_0(s) = (\gamma + \delta)s$ *and* $\tilde{\gamma}_1(s) = \delta s$*,* γ *is positive constant and* δ *is arbitrary small positive real number.*

Theorem 2 *Suppose that the strong average of system (1) exists and satisfies Assumption 1. Then, for any* $\delta > 0$ *there exists* $\varepsilon^* > 0$ *and for all* $\varepsilon \in (0, \varepsilon^*)$*,* $w \in \mathcal{L}_\infty$ *and* $x_0 := x(t_0) \in \mathbb{R}^n$*, the solution of system (1) satisfies:*

$$|x(t)| \leq (K + \delta)\exp(-(\lambda - \delta)(t - t_0))|x_0| + (\gamma + \delta)||w||_\infty \qquad \forall t \geq t_0 \geq 0, \tag{10}$$

where positive constants K, λ, γ *come from Remark 4. Thus, the system (1) is exponentially ISS uniformly in small* ε*.*

For system (2), we assume that its partial strong average satisfies the following assumption.

Assumption 2 *Consider the partial strong average $\dot{y} = A_{av}y + B_{\rho 2(t,\varepsilon)w}$ to system (2), suppose there exist positive real numbers ε^*, γ_a, c_1, c_2, and a continuously differentiable function $V(t,y)$ such that for all $\varepsilon \in (0, \varepsilon^*)$, the following holds:*

$$c_1|y|^2 \leq V(t,y) \leq c_2|y|^2, \quad \frac{\partial V}{\partial t}(t,y) + \frac{\partial V}{\partial y}(t,y)\left(Ay + B_{\rho_2(\frac{t}{\varepsilon})}w\right) \leq -|y|^2 + \gamma_a|w|^2 \quad \forall y, w. \tag{11}$$

With Assumption 2 and noting that Remark 4 also holds under this assumption, we have the following Theorem 3. Although this result applies to system (2), being a special case of system (2), we can also obtain stronger conclusions of exponential ISS properties for the actual linear switched system (1) when strong average does not exist (as opposed DISS which is a conclusion of Theorem 1) with the notion of partial strong average.

Theorem 3 *Suppose that the partial strong average of system (2) exists and satisfies Assumption 2. Then, for any $\delta > 0$ there exists $\varepsilon^* > 0$ and for all $\varepsilon \in (0, \varepsilon^*)$, $w \in \mathcal{L}_\infty$ and $x_0 \in \mathbb{R}^n$, the solution of system (2) satisfies:*

$$|x(t)| \leq (K+\delta)\exp(-(\lambda-\delta)(t-t_0))|x_0| + (\gamma+\delta)||w||_\infty \qquad \forall t \geq t_0 \geq 0. \tag{12}$$

where positive constants K, λ, γ come from Remark 4. Thus, the system (2) is exponentially ISS uniformly in small ε.

We have stated that exponential ISS of strong or partial strong average implies the actual system is exponentially ISS when the parameter ε is very small. Moreover, the linear ISS gain of the original system converges to the estimated linear ISS gain of its strong or partial strong average obtained via its Lyapunov function. On the other hand, it is impossible to prove this result for weak averages without the assumption that the disturbances are absolutely continuous. We revisit Example 1 to show that the weak average system is exponentially ISS, while the actual switched system is also exponentially ISS but with a larger linear gain. Note that the actual system coincides with its partial strong average in this example.

Example 1 *(continued): Let the switching signal be the same as Example 1. Let $A_1 = A_2 = -0.5$, $B_1 = 1$, $B_2 = -1$. Note that the weak average of system (1) is disturbance free and it is uniformly globally exponentially stable. In other words, it is exponentially ISS with zero disturbance gain. Now we consider a bounded disturbance for the actual system and we show that its linear gain can not converge to zero. Consider the following input with a constant $c > 0$:*

$$w(t) = c \;\; \textit{when} \;\; \rho(t) = 1, \qquad\qquad w(t) = -c \;\; \textit{when} \;\; \rho(t) = 2. \tag{13}$$

Then, the actual system with the given disturbance evolves like $\dot{x} = -0.5x + c$, and the solution of which satisfies $|x(t)| = \exp(-0.5(t-t_0))|x_0| + 2c$. As $||w||_\infty = c$ we get the linear gain of the actual system is lower bounded by 2, which is much lager than the ISS gain of its weak average. Hence, the results of Theorem 1 can be applied only when $\dot{w} \in \mathcal{L}_\infty$ is satisfied. In contrast, with the notion of partial strong average, we can get stronger conclusions such as exponential ISS of the actual system without requiring $\dot{w} \in \mathcal{L}_\infty$.

4 Cascaded Fast Switching Systems

The weak average is quite useful for analysis of stability properties of several classes of time-varying interconnected systems, where the input of one subsystem is the output of another subsystem. In particular, it is an important motivation for use of weak average in analysis of ISS of time-varying switched cascaded systems. Next, we will present a corollary that can be derived from Theorem 1 for the following cascaded switched system:

$$\dot{\xi} = A_{\rho(\frac{t}{\varepsilon})}\xi + B_{\rho(\frac{t}{\varepsilon})}\eta \qquad\qquad \dot{\eta} = C_{\rho(\frac{t}{\varepsilon})}\eta, \tag{14}$$

where $\xi \in \mathbb{R}^{n_1}$, $\eta \in \mathbb{R}^{n_2}$.

Corollary 1 *Suppose that the weak average of ξ-subsystem exists and is exponentially ISS with respect to η, and the average of η-subsystem is exponentially stable, then there exists a $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, the system (14) is exponentially stable uniformly in ε.*

Note that to get the exponential stability of the cascade, the ξ-subsystem does not have to be uniformly exponentially ISS with respect to η and instead we can use the exponential DISS property that was concluded in Theorem 1.

Example 2 *Consider the switched cascaded system*

$$\dot{\xi} = -0.5\xi + k_{\rho(\frac{t}{\varepsilon})}\eta \qquad\qquad \dot{\eta} = -\left(2 - k_{\rho(\frac{t}{\varepsilon})}\right)\eta, \tag{15}$$

where switching signal $\rho(\cdot)$ is the same as Example 1. Let $k_1 = 1$ and $k_2 = -1$. The weak average of the ξ-subsystem (when η is regarded as the input) was shown in Example 1 to be ISS with zero gain. Note also that the average of η-subsystem is exponentially stable and hence from Corollary 1, we conclude that the cascaded system (15) is exponentially stable.

Remark 6 *Related results were presented in [7] where $\mathcal{L}_2$ stability of rapidly switching linear systems was considered. In particular, [7] shows that if the input matrix B does not switch, the $\mathcal{L}_2$ gain of the actual time varying switched system is bounded by the $\mathcal{L}_2$ gain of its average as the switching rate* *is increased. We consider a different stability property (ISS as apposed to $\mathcal{L}_2$ stability), and we get stronger conclusion that the linear ISS gain can be recovered for linear switched systems that allow for strong averages and partial strong averages (see Theorems 2 and 3). Moreover, [7] shows via an example that if the input matrix B switches, then the $\mathcal{L}_2$ gain of the actual switched system may not be bounded by the $\mathcal{L}_2$ gain of its average when the switching rate increases. On the other hand, we show in Theorem 1 that the ISS gain of the actual system also can be recovered by its weak average (note that δ in (9) can be arbitrarily small when ε is sufficiently small) if we restrict the derivatives of disturbances to be uniformly bounded. We also show that the linear ISS gain of the actual system converges to the ISS gain of its partial strong average without requiring the derivatives of disturbances to be bounded in Theorem 3.*

5 Conclusions

ISS properties of linear rapidly switching systems with disturbances via the averaging method were investigated. With the notions of strong and weak averages pioneered in [1] and partial strong average that was proposed in the present paper, several averaging results were derived. We proved that exponential ISS of the strong and the partial strong average system implies exponential ISS for the actual linear switched system with their linear gains converging to each other as the parameter is reduced. Moreover, exponential ISS of the weak average guarantees an appropriate DISS property for the actual system. We emphasize that one contribution of this paper is a systematic use of strong, partial strong and weak averages for switched systems with disturbances that we believe will be very useful in a range of other averaging questions for switched systems.

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Appendix

We start from the technical lemma that would be useful for proving Theorem 1.

Lemma 1 *Suppose that the weak average of system (1) exists and satisfies Assumption 1 with Lyapunov function V and positive constants c_1, c_2 and γ_a. Then, for any $\tilde{\delta} \in (0,1)$ there exists $\tilde{\tau}^* > 0$ such that, for each $\tau \in (0, \tilde{\tau}^*)$ there exist $\varepsilon^* > 0$ and an increasing sequence of times $t_i (i \in \mathbb{N}) : t_{i+1} - t_i \leq \tau$ with $t_i \to \infty$ as $i \to \infty$, such that for all $t_i \geq t_0$, $\varepsilon \in (0, \varepsilon^*)$, $w, \dot{w} \in \mathcal{L}_\infty$ and $x_0 := x(t_0) \in \mathbb{R}^n$, the solution of system (1) satisfies:*

$$\frac{V(x(t_{i+1})) - V(x(t_i))}{\tau} \leq -\left(1 - \tilde{\delta}\right) |x(t_i)|^2 + \left(\gamma_a + \tilde{\delta}\right) ||w||_\infty^2 + \tilde{\delta} ||\dot{w}||_\infty^2. \tag{16}$$

Proof of Lemma 1: Let a positive real number $\tilde{\delta} < 1$ be given. Let the quadratic Lyapunov function V and positive constants c_1, c_2 and γ_a come from Assumption 1. Let $k_1, k_2, T^* > 0$ and $\sigma_1, \sigma_2 \in \mathcal{L}$ come from the definition of average for matrices A_ρ and B_ρ. In preparation for defining ε^*, let $\tilde{T}_1 \geq T^*$ and $\tilde{T}_2 \geq T^*$ satisfying $\sigma_1(\tilde{T}_1) \leq \frac{\tilde{\delta}}{8k_1c_2}$, $\sigma_2(\tilde{T}_2) \leq \frac{\tilde{\delta}}{8k_2c_2}$, and define $\tilde{T} = \max\{\tilde{T}_1, \tilde{T}_2\}$. Let a strictly positive real number $a_m = \max_{t \geq t_0}\{|A_{\rho(t)}|, |B_{\rho(t)}|\}$ where $|\cdot|$ denotes the matrix norm induced by the vector norm. Let τ_1 satisfies $\tau_1 \leq \frac{\tilde{\delta}}{8c_2a_m}$ and $\tau_2 > 0$ be such that for the given $\tilde{\delta}$, for any $t_i \geq t_0$ and positive constant $\tilde{k}$ the following holds: $(s - t_i) \in [0, \tau_2] \Rightarrow \tilde{k}(s - t_i)\exp(2a_m(s - t_i)) \leq \tilde{\delta}$. Then define $\tilde{\tau}^* := \min\{\tau_1, \tau_2\}$. For all $\tau \in (0, \tilde{\tau}^*)$, let $\varepsilon^* := \left\{\frac{\tau}{\tilde{T}}\right\}$, and $t_i = t_0 + i\varepsilon\tilde{T}$ for all $\varepsilon \in (0, \varepsilon^*)$ and $i \in \mathbb{N}$. From the definition of ε^*, we have $t_{i+1} - t_i = \varepsilon\tilde{T} \leq \tau$. Letting $x(t) := x(t, t_i)$ and applying the

Lyapunov candidate function V in Assumption 1 to system (1) for all $t \in [t_i, t_{i+1}]$, it follows that

$$\begin{aligned}
&\frac{\partial V}{\partial x}(x(t))\{A_{\rho(\frac{t}{\varepsilon})}x(t) + B_{\rho(\frac{t}{\varepsilon})}w(t)\} \\
&\quad = \frac{\partial V}{\partial x}(x(t_i))\{A_{av}x(t_i) + B_{av}w(t)\} - \frac{\partial V}{\partial x}(x(t_i))\{B_{av}w(t) - B_{av}w(t_i)\} \\
&\quad - \frac{\partial V}{\partial x}(x(t_i))\{A_{av}x(t_i) + B_{av}w(t_i)\} + \frac{\partial V}{\partial x}(x(t_i))\{A_{\rho(\frac{t}{\varepsilon})}x(t_i) + B_{\rho(\frac{t}{\varepsilon})}w(t_i)\} \\
&\quad + \frac{\partial V}{\partial x}(x(t))\{A_{\rho(\frac{t}{\varepsilon})}x(t) + B_{\rho(\frac{t}{\varepsilon})}w(t)\} - \frac{\partial V}{\partial x}(x(t_i))\{A_{\rho(\frac{t}{\varepsilon})}x(t_i) + B_{\rho(\frac{t}{\varepsilon})}w(t_i)\}.
\end{aligned} \tag{17}$$

Integrating both sides of the inequality along the solution of $x(t)$ over the interval $[t_i, t_{i+1}]$ and with the fact $\left|\frac{\partial V}{\partial x}(x(t_i))\right| \leq 2c_2|x(t_i)|$, we get

$$\begin{aligned}
\frac{V(x(t_{i+1})) - V(x(t_i))}{\varepsilon\tilde{T}} &\leq \underbrace{\frac{1}{\varepsilon\tilde{T}}\int_{t_i}^{t_{i+1}}\frac{\partial V}{\partial x}(x(t_i))\{A_{av}x(t_i) + B_{av}w(s)\}ds}_{1} + \underbrace{\frac{2c_2|x(t_i)|}{\varepsilon\tilde{T}}\left|\int_{t_i}^{t_{i+1}}B_{av}(w(s) - w(t_i))ds\right|}_{2} \\
&+ \underbrace{\frac{2c_2|x(t_i)|}{\varepsilon\tilde{T}}\left\{\left|\int_{t_i}^{t_{i+1}}\{A_{av} - A_{\rho(\frac{s}{\varepsilon})}\}x(t_i)ds\right| + \left|\int_{t_i}^{t_{i+1}}\{B_{av} - B_{\rho(\frac{s}{\varepsilon})}\}w(t_i)ds\right|\right\}}_{3} \\
&+ \underbrace{\frac{1}{\varepsilon\tilde{T}}\left|\int_{t_i}^{t_{i+1}}\left\{\frac{\partial V}{\partial x}(x(s))A_{\rho(\frac{s}{\varepsilon})}x(s) + \frac{\partial V}{\partial x}(x(s))B_{\rho(\frac{s}{\varepsilon})}w(s) - \frac{\partial V}{\partial x}(x(t_i))\left(A_{\rho(\frac{s}{\varepsilon})}x(t_i) + B_{\rho(\frac{s}{\varepsilon})}w(t_i)\right)\right\}ds\right|}_{4}
\end{aligned} \tag{18}$$

We now turn to bounding each of the terms on the right-hand side of (18), and the inequality $2ab \leq a^2 + b^2$ is utilized in the following proof.

1. From Assumption 1, it follows that the term 1 is bounded by

$$\tfrac{1}{\varepsilon\tilde{T}}\textstyle\int_{t_i}^{t_{i+1}}\{-|x(t_i)|^2 + \gamma_a|w(s)|^2\}ds \leq -|x(t_i)|^2 + \gamma_a||w||_\infty^2.$$

2. With the definition of τ, we have for all $t \in [t_i, t_{i+1}]$ that term 2 is bounded by

$$2c_2a_m\tau|x(t_i)| \cdot ||\dot{w}||_\infty \leq \frac{\tilde{\delta}}{8}(|x(t_i)|^2 + ||\dot{w}||_\infty^2).$$

3. Setting $s = \varepsilon\nu$ and considering $A_{\rho(\frac{s}{\varepsilon})}$, $B_{\rho(\frac{s}{\varepsilon})}$ and the average definition for matrices, we have that term 3 can be bounded by

$$\begin{aligned}
&2c_2|x(t_i)|\left\{\left|A_{av} - \frac{1}{\tilde{T}}\int_{\frac{t_i}{\varepsilon}}^{\frac{t_i}{\varepsilon}+\tilde{T}}A_{\rho(\nu)}d\nu\right| \cdot |x(t_i)| + \left|B_{av} - \frac{1}{\tilde{T}}\int_{\frac{t_i}{\varepsilon}}^{\frac{t_i}{\varepsilon}+\tilde{T}}B_{\rho(\nu)}d\nu\right| \cdot ||w||_\infty\right\} \\
&\leq 2c_2|x(t_i)|\{k_1\sigma_1(\tilde{T})|x(t_i)| + k_2\sigma_2(\tilde{T})||w||_\infty\} \leq \tfrac{3\tilde{\delta}}{8}|x(t_i)|^2 + \tfrac{\tilde{\delta}}{8}||w||_\infty^2.
\end{aligned}$$

Finally, term 4 is bounded by

$$\begin{aligned}
&\left|\frac{\partial V}{\partial x}(x(t))\left\{A_{\rho(\frac{t}{\varepsilon})}(x(t) - x(t_i)) + B_{\rho(\frac{t}{\varepsilon})}(w(t) - w(t_i))\right\}\right| + \left|\left\{\frac{\partial V}{\partial x}(x(t)) - \frac{\partial V}{\partial x}(x(t_i))\right\}\{A_{\rho(\frac{t}{\varepsilon})}x(t_i) + B_{\rho(\frac{t}{\varepsilon})}w(t_i)\}\right| \\
&\leq \quad 2c_2a_m\{|x(t)| \cdot (|x(t) - x(t_i)| + |w(t) - w(t_i)|) + |x(t) - x(t_i)|(|x(t_i)| + ||w||_\infty)\}.
\end{aligned} \tag{19}$$

As for all $t \in [t_i, t_{i+1}]$ and $t_i \geq t_0$, the solution of system (1) satisfies $x(t) = x(t_i) + \int_{t_i}^{t} \left(A_{\rho(\frac{s}{\varepsilon})} x(s) + B_{\rho(\frac{s}{\varepsilon})} w(s) \right) ds$. With the definition of a_m, it is obvious that for all $t \in [t_i, t_{i+1}]$

$$|x(t)| \leq \exp(a_m \tau)|x(t_i)| + (\exp(a_m \tau) - 1)||w||_\infty. \tag{20}$$

Noting $|w(t) - w(t_i)| \leq ||\dot{w}||_\infty \tau$ and $|x(t) - x(t_i)| \leq a_m \tau(|x(t)| + ||w||_\infty)$, it follows that the inequality (19) is bounded by $2\tau \exp(2a_m\tau)c_2\{5a_m^2(|x(t_i)|^2 + ||w||_\infty^2) + a_m||\dot{w}||_\infty^2\}$. Letting $\tilde{k} := 4c_2(10a_m^2 + a_m)$, noting the definition of τ_2 and $\tilde{k} \geq \frac{4c_2\{5a_m^2(|x(t_i)|^2 + ||w||_\infty^2) + a_m||\dot{w}||_\infty^2\}}{\max\{|x(t_i)|^2, ||w||_\infty^2, ||\dot{w}||_\infty^2\}}$ holds for all $x(t_i)$ and w , it follows that term 4 is bounded by $\frac{\tilde{\delta}}{2}(|x(t_i)|^2 + ||w||_\infty^2 + ||\dot{w}||_\infty^2)$ for all $\tau \in (0, \tilde{\tau}^*)$. Combining the upper bound of four terms in (18), we complete the proof:

$$\begin{aligned} \frac{V(x(t_{i+1})) - V(x(t_i))}{\tau} &\leq -\left(1 - \tilde{\delta}\right)|x(t_i)|^2 + \left(\gamma_a + \frac{5\tilde{\delta}}{8}\right)||w||_\infty^2 + \frac{5\tilde{\delta}}{8}||\dot{w}||_\infty^2 \\ &\leq -\left(1 - \tilde{\delta}\right)|x(t_i)|^2 + \left(\gamma_a + \tilde{\delta}\right)||w||_\infty^2 + \tilde{\delta}||\dot{w}||_\infty^2. \end{aligned}$$

Proof of Theorem 1: Let the quadratic Lyapunov function V and positive constants c_1, c_2 and γ_a come from Assumption 1, K, λ and γ come from Remark 4. Given any $0 < \tilde{\delta} < 1$, $\tilde{\tau}^*$ is then determined from Lemma 1 by $\tilde{\delta}$. Let a_m be the same as the proof of Lemma 1 and $c := 1 - \tilde{\delta} > 0$. Let $\tau_3 = \frac{c_2}{c}$ and $\tau^* = \min\{\tilde{\tau}^*, \tau_3\}$, then we have $0 < \left(1 - \frac{c\tau}{c_2}\right) < 1$ when $\tau \in (0, \tau^*)$. Let ε^* be determined by τ from Lemma 1, and the proof is followed for all $\varepsilon \in (0, \varepsilon^*)$. For preparing the definition of δ, let $m := 1 - \frac{c\tau}{c_2}$, $\quad \mu := \sqrt{\frac{\tau}{c_1(1-m)}}$. Then, let

$$\begin{aligned} \delta_1 &= (\exp(a_m\tau) - 1)(\gamma + K + 1) + \exp(a_m\tau)\left(\gamma\sqrt{\frac{\tilde{\delta}}{1 - \tilde{\delta}}} + \mu\sqrt{\tilde{\delta}}\right), \\ \delta_2 &= \frac{\tilde{\delta}}{2c_2},\ \delta_3 = \left(\exp\left(2\tau\left(a_m + \lambda - \frac{\tilde{\delta}}{2c_2}\right)\right) - 1\right)K,\ \delta := \max\{\delta_1, \delta_2, \delta_3, \tilde{\delta}\}. \end{aligned} \tag{21}$$

With a_m defined above, for all $t \geq t_0$, solutions of system (1) satisfy $|x(t)| \leq \exp(a_m(t - t_0))|x_0| + (\exp(a_m(t - t_0)) - 1)||w||_\infty$, and then there exists a t_{i_0} such that $t_{i_0} - t_0 \leq \tau$ implies

$$|x(t_{i_0})| \leq \exp(a_m\tau)|x_0| + (\exp(a_m\tau) - 1)||w||_\infty. \tag{22}$$

From Lemma 1, for all $\varepsilon \in (0, \varepsilon^*)$ and any t_{i_0} satisfying $t_{i_0} - t_0 \leq \tau$ and $t_{i_0+k} - t_{i_0+k-1} \leq \tau$, $\forall k \in \mathbb{N}$, we have

$$\tfrac{V(x(t_{i_0+1})) - V(x(t_{i_0}))}{\tau} \leq -c|x(t_{i_0})|^2 + \left(\gamma_a + \tilde{\delta}\right)||w||_\infty^2 + \tilde{\delta}||\dot{w}||_\infty^2.$$

Using $V(x) \leq c_2|x|^2$ and noting $m = 1 - \frac{c\tau}{c_2}$, it follows that

$$V(x(t_{i_0+1})) \leq mV(x(t_{i_0})) + \tau\left(\gamma_a + \tilde{\delta}\right)||w||_\infty^2 + \tau\tilde{\delta}||\dot{w}||_\infty^2.$$

By repeating this argument, and using $(1 - z)^n \leq \exp(-nz)$, $\forall z \in (0, 1)$, we have $\forall n \in \mathbb{N}$

$$\begin{aligned} V(x(t_{i_0+n})) &\leq m^n V(x(t_{i_0})) + \textstyle\sum_{k=1}^n m^{k-1}\tau(\gamma_a + \tilde{\delta})||w||_\infty^2 + \sum_{k=1}^n m^{k-1}\tau\tilde{\delta}||\dot{w}||_\infty^2 \\ &\leq \exp\left(-\tfrac{c\tau n}{c_2}\right)V(x(t_{i_0})) + \tfrac{\tau(\gamma_a + \tilde{\delta})}{(1-m)}||w||_\infty^2 + \tfrac{\tau\tilde{\delta}}{(1-m)}||\dot{w}||_\infty^2. \end{aligned}$$

From $c_1|x|^2 \leq V(x) \leq c_2|x|^2$, we know that

$$|x(t_{i_0+n})|^2 \leq \frac{c_2}{c_1}\exp\left(-\frac{c\tau n}{c_2}\right)|x(t_{i_0})|^2 + \frac{\tau(\gamma_a+\tilde{\delta})}{c_1(1-m)}||w||_\infty^2 + \frac{\tau\tilde{\delta}}{c_1(1-m)}||\dot{w}||_\infty^2.$$

With the definition of μ and K, and noting $t_{i_0+n} - t_{i_0} \leq n\tau$, it follows that

$$|x(t_{i_0+n})| \leq K\exp\left(-\tfrac{c}{2c_2}(t_{i_0+n}-t_{i_0})\right)|x(t_{i_0})| + \mu\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right)||w||_\infty + \mu\sqrt{\tilde{\delta}}||\dot{w}||_\infty. \tag{23}$$

Letting $\bar{\lambda} = \frac{c}{2c_2}$, and by repeating the same argument, one knows that for every $j \in \mathbb{N}$

$$|x(t_{i_0+jn})| \leq K\exp\left(-\bar{\lambda}(t_{i_0+jn}-t_{i_0})\right)|x(t_{i_0})| + \mu\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right)||w||_\infty + \mu\sqrt{\tilde{\delta}}||\dot{w}||_\infty. \tag{24}$$

For every $t \geq t_{i_0}$, there exist $j \in \mathbb{N}$ such that $t \in [t_{i_0+jn}, t_{i_0+jn+1})$. While (24) gives the evolution of the flow at times t_{i_0+jn}, $\forall j \in \mathbb{N}$, but gives no information about the flow between the times t_{i_0+jn}. Considering (20), it follows that $|x(t)| \leq \exp(a_m\tau)|x(t_{i_0+jn})| + (\exp(a_m\tau)-1)||w||_\infty$, $\forall t \in [t_{i_0+jn}, t_{i_0+jn+1})$. Noting (24), it follows that for all $t \geq t_{i_0}$, the solution of system (1) satisfies

$$\begin{aligned} |x(t)| &\leq \exp(a_m\tau)K\exp(-\bar{\lambda}(t_{i_0+jn}-t_{i_0}))|x(t_{i_0})| + (\exp(a_m\tau)-1)||w||_\infty \\ &\quad +\mu\exp(a_m\tau)\sqrt{\tilde{\delta}}||\dot{w}||_\infty + \mu\exp(a_m\tau)\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right)||w||_\infty \\ &\leq \exp((a_m+\bar{\lambda})\tau)K\exp\left(-\bar{\lambda}(t-t_{i_0})\right)|x(t_{i_0})| + \mu\exp(a_m\tau)\sqrt{\tilde{\delta}}||\dot{w}||_\infty \\ &\quad + (\exp(a_m\tau)-1)\,||w||_\infty + \mu\exp(a_m\tau)\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right)||w||_\infty. \end{aligned} \tag{25}$$

With $\gamma = \sqrt{\frac{c_2\gamma_a}{c_1}}$, the definitions of m and δ, we have $\mu\sqrt{\gamma_a} = \sqrt{\frac{c_2\gamma_a}{c_1(1-\tilde{\delta})}} = \gamma\cdot\sqrt{1+\frac{\tilde{\delta}}{1-\tilde{\delta}}}$ and

$$\begin{aligned} &\mu\exp(a_m\tau)\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right) + (\exp(a_m\tau)-1)(K+1) \\ &\quad\leq \exp(a_m\tau)\left(\gamma+\gamma\sqrt{\tfrac{\tilde{\delta}}{1-\tilde{\delta}}}+\mu\sqrt{\tilde{\delta}}\right) + (\exp(a_m\tau)-1)(K+1) \leq \gamma+\delta. \end{aligned} \tag{26}$$

Considering the definition of δ, we know that

$$\exp(2(a_m+\bar{\lambda})\tau)K = \exp\left(2\left(a_m+\lambda-\frac{\tilde{\delta}}{2c_2}\right)\tau\right)K \leq K+\delta. \tag{27}$$

Then, combining $\bar{\lambda} = \frac{1-\tilde{\delta}}{2c_2} \geq \lambda-\delta$, (21), (22) and (26) into (25), the following completes the proof:

$$\begin{aligned} |x(t)| &\leq \exp(2(a_m+\bar{\lambda})\tau)K\exp\left(-\bar{\lambda}(t-t_0)\right)|x_0| + \mu\exp(a_m\tau)\left(\sqrt{\gamma_a}+\sqrt{\tilde{\delta}}\right)||w||_\infty \\ &\quad + (\exp(a_m\tau)-1)\,(K+1)||w||_\infty + \exp(a_m\tau)\mu\sqrt{\tilde{\delta}}||\dot{w}||_\infty \\ &\leq (K+\delta)(\exp(-(\lambda-\delta)(t-t_0))|x_0| + (\gamma+\delta)||w||_\infty + \delta||\dot{w}||_\infty \end{aligned}$$